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AN ALGEBRAIC OPERATIONS FOR TWO GENERALIZED 2-DIMENSIONAL QUADRATIC FUZZY SETS

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Abstract. We generalized the quadratic fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$. By defining parametric operations between two regions valued α-cuts, we got the parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$. The results for the parametric operations are the generalization of Zadeh's extended algebraic operations. We generalize the 2-dimensional quadratic fuzzy numbers on $\mathbb{R}^2$ that may have maximum value $h < 1$. We calculate the algebraic operations for two generalized 2-dimensional quadratic fuzzy sets.

1. Introduction

We calculated the algebraic operator for two generalized trapezoidal fuzzy sets ([6]) and for two one-sided quadrangular fuzzy sets ([7]). We generalized the triangular fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$ and calculated the algebraic operator for two 2-dimensional triangular fuzzy numbers ([4]). We proved that the results for the parametric operations are the generalization of Zadeh's max-min composition operations ([2]).

We generalized the quadratic fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$ ([3]). By defining parametric operations between two regions valued α-cuts, we got the parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$. The results for the parametric operations are the generalization of Zadeh's extended algebraic operations ([5]). We generalize the 2-dimensional quadratic fuzzy numbers on $\mathbb{R}^2$ that may have maximum value $h < 1$. We calculate the algebraic operations for two generalized 2-dimensional quadratic fuzzy sets.

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2. Preliminaries

We define α-cut and α-set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$.

Definition 2.1. An *α-cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_0 = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$. For $\alpha \in (0,1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ and $A^1 = A_1$.

Following Zadeh, Dubois and Prade, the extension principle is defined as follows:

Definition 2.2. [8] The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$ and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z=x*y} \min\{\mu_A(x), \mu_B(y)\},\ * = +, -, \cdot, /.$$

We defined the parametric operations for two fuzzy numbers defined on $\mathbb{R}$ and showed that the results for parametric operations are the same as those for the extended operations ([1]). For this, we proved that for all fuzzy numbers A and all $\alpha \in [0,1]$, there exists a piecewise continuous function $f_\alpha(t)$ defined on $[0,1]$ such that $A_\alpha = \{f_\alpha(t) | t \in [0,1]\}$. If A is continuous, then the corresponding function $f_\alpha(t)$ is also continuous. The corresponding function $f_\alpha(t)$ is said to be the *parametric α-function* of A. The parametric α-function of A is denoted by $f_\alpha(t)$ or $f_A(t)$.

Definition 2.3. Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$ and $f_A(t), f_B(t)$ be the parametric α-functions of A and B, respectively. The parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers that have their α-cuts as follows.

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)_\alpha = \{f_A(t) + f_B(t) \mid t \in [0,1]\}.$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)_\alpha = \{f_A(t) - f_B(1-t) \mid t \in [0,1]\}.$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)_\alpha = \{f_A(t) \cdot f_B(t) \mid t \in [0,1]\}.$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)_\alpha = \{f_A(t)/f_B(1-t) \mid t \in [0,1]\}.$$

THEOREM 2.4. [1] *Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$. Then we have $A(+)_pB = A(+)B, A(-)_pB = A(-)B, A(\cdot)_pB = A(\cdot)B$ and $A(/)_pB = A(/)B$.*

3. A generalized 2-dimensional quadratic fuzzy sets

In this section, we define the generalized 2-dimensional quadratic fuzzy sets on $\mathbb{R}^2$. We defined the parametric operations between two 2-dimensional quadratic fuzzy numbers using the operations between α-sets in $\mathbb{R}^2$. In $\mathbb{R}^2$ the α-sets are regions, which makes the existing method of calculations between α-sets. We interpret the existing method from a different perspective and apply the method to the region valued α-sets on $\mathbb{R}^2$.

DEFINITION 3.1. A fuzzy set A with a membership function $\mu_A(x,y)$

$$= \begin{cases} h - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}\right), & b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq ha^2b^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b > 0$ and $0 < h < 1$ is called the *the generalized 2-dimensional quadratic fuzzy set* and denoted by $[[a,\ x_1,\ h,\ b,\ y_1]]^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha \quad (0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1) \quad (k \in \mathbb{R})$ are symmetric quadratic fuzzy sets in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a generalized 2-dimensional quadratic fuzzy set $A = [[a, x_1, h, b, y_1]]^2$ is an interior of ellipse in an xy-plane including the boundary

$$\begin{aligned} A_\alpha &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2(h-\alpha)\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{(x-x_1)^2}{a^2(h-\alpha)} + \frac{(y-y_1)^2}{b^2(h-\alpha)} \leq 1\right\}. \end{aligned}$$

THEOREM 3.2. [1] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}.$$

DEFINITION 3.3. Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$
$$B^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_B(x,y) = \alpha\} = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition $A(+)_pB$, parametric subtraction $A(-)_pB$, parametric multiplication $A(\cdot)_pB$ and parametric division $A(/)_pB$ of two fuzzy numbers A and B are fuzzy numbers that have their α-sets as follows.

(1) $A(+)_pB$:
$(A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$.

(2) $A(-)_pB$:
$(A(-)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi. \end{cases}$$

(3) $A(\cdot)_pB$:
$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$.

(4) $A(/)_pB$:
$(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi),$$

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi),$$

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi).$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha \to 0^+} (A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha \to 1^-} (A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 3.4. *Let $A = [[a_1,\ x_1,\ h_1,\ b_1,\ y_1]]^2$ and $B = [[a_2,\ x_2,\ h_2,\ b_2,\ y_2]]^2$ $(0 < h_1 < h_2 < 1)$ be two generalized 2-dimensional quadratic fuzzy sets. Since A and B are convex fuzzy sets defined on $\mathbb{R}^2$, by Theorem 3.2, there exists $f_i^\alpha(t), g_i^\alpha(t)$ $(i = 1, 2)$ such that*

$$A^\alpha = \{(x, y) \in \mathbb{R}^2 | \mu_A(x, y) = \alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}$$

and

$$B^\alpha = \{(x, y) \in \mathbb{R}^2 | \mu_B(x, y) = \alpha\} = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}.$$

Since $A = [[a_1,\ x_1,\ h_1,\ b_1,\ y_1]]^2$ and $B = [[a_2,\ x_2,\ h_2,\ b_2,\ y_2]]^2$, we have

$$f_1^\alpha(t) = x_1 + a_1\sqrt{h_1 - \alpha}\cos t,\quad f_2^\alpha(t) = y_1 + b_1\sqrt{h_1 - \alpha}\sin t$$

and

$$g_1^\alpha(t) = x_2 + a_2\sqrt{h_2 - \alpha}\cos t,\quad g_2^\alpha(t) = y_2 + b_2\sqrt{h_2 - \alpha}\sin t.$$

We have the followings.

(1) *Let $0 < \alpha < h_1$. Since*

$$f_1^\alpha(t) + g_1^\alpha(t) = x_1 + x_2 + (a_1\sqrt{h_1 - \alpha} + a_2\sqrt{h_2 - \alpha})\cos t$$

and

$$f_2^\alpha(t) + g_2^\alpha(t) = y_1 + y_2 + (b_1\sqrt{h_1 - \alpha} + b_2\sqrt{h_2 - \alpha})\sin t,$$

we have

$$(A(+)_pB)^\alpha = \Big\{(x, y) \in \mathbb{R}^2 \Big| \Big(\frac{x - x_1 - x_2}{a_1\sqrt{h_1 - \alpha} + a_2\sqrt{h_2 - \alpha}}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1\sqrt{h_1 - \alpha} + b_2\sqrt{h_2 - \alpha}}\Big)^2 = 1\Big\}.$$

(2) *Let $0 < \alpha < h_1$. If $0 \leq t \leq \pi$,*

$$f_1^\alpha(t) - g_1^\alpha(t + \pi) = x_1 - x_2 + (a_1\sqrt{h_1 - \alpha} + a_2\sqrt{h_2 - \alpha})\cos t$$

and

$$f_2^\alpha(t) - g_2^\alpha(t + \pi) = y_1 - y_2 + (b_1\sqrt{h_1 - \alpha} + b_2\sqrt{h_2 - \alpha})\sin t.$$

In the case of $\pi \leq t \leq 2\pi$, we have

$$f_1^\alpha(t) - g_1^\alpha(t - \pi) = f_1^\alpha(t) - g_1^\alpha(t + \pi)$$

and

$$f_2^\alpha(t) - g_2^\alpha(t - \pi) = f_2^\alpha(t) - g_2^\alpha(t + \pi).$$

Thus

$$(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \Big| \Big(\frac{x - x_1 + x_2}{a_1\sqrt{h_1-\alpha} + a_2\sqrt{h_2-\alpha}}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_1\sqrt{h_1-\alpha} + b_2\sqrt{h_2-\alpha}}\Big)^2 = 1\Big\}.$$

(3) *Let* $0 < \alpha < h_1$ *and* $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. *Since*

$f_1^\alpha(t) = x_1 + a_1\sqrt{h_1-\alpha}\cos t,\ \ f_2^\alpha(t) = y_1 + b_1\sqrt{h_1-\alpha}\sin t$

and

$g_1^\alpha(t) = x_2 + a_2\sqrt{h_2-\alpha}\cos t,\ \ g_2^\alpha(t) = y_2 + b_2\sqrt{h_2-\alpha}\sin t,$

we have

$$x_\alpha(t) = x_1x_2 + (x_1a_2\sqrt{h_2-\alpha} + x_2a_1\sqrt{h_1-\alpha})\cos t + a_1a_2\sqrt{h_1-\alpha}\sqrt{h_2-\alpha}\cos^2 t$$

and

$$y_\alpha(t) = y_1y_2 + (y_1b_2\sqrt{h_2-\alpha} + y_2b_1\sqrt{h_1-\alpha})\sin t + b_1b_2\sqrt{h_1-\alpha}\sqrt{h_2-\alpha}\sin^2 t.$$

(4) *Let* $0 < \alpha < h_1$ *and* $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. *Similarly, we have*

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{h_1-\alpha}\cos t}{x_2 - a_2\sqrt{h_2-\alpha}\cos t} \quad \text{and}$$

$$y_\alpha(t) = \frac{y_1 + b_1\sqrt{h_1-\alpha}\sin t}{y_2 - b_2\sqrt{h_2-\alpha}\sin t}.$$

If $\alpha = h_1$, *we have* $(A(*)_pB)^{h_1} = \lim_{\alpha\to h_1^-}(A(*)_pB)^\alpha$, $* = +,\ -,\ \cdot,\ /$, *and for* $h_1 < \alpha \le h_2$, *by the Zadehs max-min principle operations, we have to define* $(A(*)_pB)^\alpha = \emptyset$, $* = +,\ -,\ \cdot,\ /$.

EXAMPLE 3.5. *Let* $A = [[6,\ 3,\ \frac{1}{2},\ 8,\ 5]]^2$ *and* $B = [[4,\ 2,\ \frac{2}{3},\ 5,\ 3]]^2$. *Then by Theorem 3.5, we have the following.*

(1) *For* $0 < \alpha < \frac{1}{2}$, *the* α*-set* $(A(+)_pB)^\alpha$ *of* $A(+)_pB$ *is*

$$\Big\{(x,y) \in \mathbb{R}^2 \Big| \Big(\frac{3(x-5)}{9\sqrt{2-4\alpha} + 4\sqrt{6-9\alpha}}\Big)^2 + \Big(\frac{3(y-8)}{12\sqrt{2-4\alpha} + 5\sqrt{6-9\alpha}}\Big)^2 = 1\Big\}.$$

(2) *For* $0 < \alpha < \frac{1}{2}$, *the* α*-set* $(A(-)_p B)^\alpha$ *of* $A(-)_p B$ *is*

$$\Big\{(x,y)\in\mathbb{R}^2\Big|\Big(\frac{3(x-1)}{9\sqrt{2-4\alpha}+4\sqrt{6-9\alpha}}\Big)^2 + \Big(\frac{3(y-2)}{12\sqrt{2-4\alpha}+5\sqrt{6-9\alpha}}\Big)^2 = 1\Big\}.$$

(3) *For* $0 < \alpha < \frac{1}{2}$, $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = 6 + (4\sqrt{6-9\alpha} + 6\sqrt{2-4\alpha})\cos t + 4\sqrt{(2-4\alpha)(6-9\alpha)}\cos^2 t,$$

$$y_\alpha(t) = 15 + (\frac{25}{3}\sqrt{6-9\alpha} + 12\sqrt{2-4\alpha})\sin t + \frac{20}{3}\sqrt{(2-4\alpha)(6-9\alpha)}\sin^2 t.$$

(4) *For* $0 < \alpha < \frac{1}{2}$, $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{9+9\sqrt{2-4\alpha}\cos t}{6-4\sqrt{6-9\alpha}\cos t}, \quad y_\alpha(t) = \frac{15+12\sqrt{2-4\alpha}\sin t}{9-5\sqrt{6-9\alpha}\sin t}.$$

For $\alpha > \frac{1}{2}$, *we have* $(A(*)_p B)^\alpha = \emptyset$, $\quad * = +,\ -,\ \cdot,\ /$.

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